Bifurcation Types
Contents

Articles

Bifurcation theory 1

Local 4

Saddle-node bifurcation 4
Transcritical bifurcation 5
Pitchfork bifurcation 6
Period-doubling bifurcation 7
Hopf bifurcation 9

Global 13

Homoclinic bifurcation 13
Heteroclinic bifurcation 14
Infinite-period bifurcation 14
Blue sky catastrophe 15

References

Article Sources and Contributors 16
Image Sources, Licenses and Contributors 17

Article Licenses

License 18
Bifurcation theory

**Bifurcation theory** is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour.[1] Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs), and discrete systems (described by maps). The name "bifurcation" was first introduced by Henri Poincaré in 1885 in the first paper in mathematics showing such a behavior.[2] Henri Poincaré also later named various types of stationary points and classified them.

**Bifurcation types**

It is useful to divide bifurcations into two principal classes:

- Local bifurcations, which can be analysed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds; and
- Global bifurcations, which often occur when larger invariant sets of the system 'collide' with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria (fixed points).

**Local bifurcations**

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is *non-hyperbolic* at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighbourhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local').

More technically, consider the continuous dynamical system described by the ODE

\[ \dot{x} = f(x, \lambda) \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \]
A local bifurcation occurs at \((x_0, \lambda_0)\) if the Jacobian matrix \(d f_{x_0, \lambda_0}\) has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

For discrete dynamical systems, consider the system

\[
x_{n+1} = f(x_n, \lambda) .
\]

Then a local bifurcation occurs at \((x_0, \lambda_0)\) if the matrix \(d f_{x_0, \lambda_0}\) has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation. If the eigenvalue is equal to \(-1\), it is a period-doubling (or flip) bifurcation, and otherwise, it is a Hopf bifurcation.

Examples of local bifurcations include:

- Saddle-node (fold) bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Period-doubling (flip) bifurcation
- Hopf bifurcation
- Neimark (secondary Hopf) bifurcation

**Global bifurcations**

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence 'global').

Examples of global bifurcations include:

- Homoclinic bifurcation in which a limit cycle collides with a saddle point.
- Heteroclinic bifurcation in which a limit cycle collides with two or more saddle points.
- Infinite-period bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.
- Blue sky catastrophe in which a limit cycle collides with a nonhyperbolic cycle.

Global bifurcations can also involve more complicated sets such as chaotic attractors (e.g. crises).

**Codimension of a bifurcation**

The codimension of a bifurcation is the number of parameters which must be varied for the bifurcation to occur. This corresponds to the codimension of the parameter set for which the bifurcation occurs within the full space of parameters. Saddle-node bifurcations and Hopf bifurcations are the only generic local bifurcations which are really codimension-one (the others all having higher codimension). However, often transcritical and pitchfork bifurcations are also often thought of as codimension-one, because the normal forms can be written with only one parameter.

An example of a well-studied codimension-two bifurcation is the Bogdanov–Takens bifurcation.

**Applications in semiclassical and quantum physics**

Bifurcation theory has been applied to connect quantum systems to the dynamics of their classical analogues in atomic systems,[3][4][5] molecular systems,[6] and resonant tunneling diodes.[7] Bifurcation theory has also been applied to the study of laser dynamics[8] and a number of theoretical examples which are difficult to access experimentally such as the kicked top[9] and coupled quantum wells.[10] The dominant reason for the link between quantum systems and bifurcations in the classical equations of motion is that at bifurcations, the signature of classical orbits becomes large, as Martin Gutzwiller points out in his classic[11] work on quantum chaos.[12] Many
kinds of bifurcations have been studied with regard to links between classical and quantum dynamics including saddle node bifurcations, Hopf bifurcations, umbilic bifurcations, period doubling bifurcations, reconnection bifurcations, tangent bifurcations, and cusp bifurcations.

Notes


References

- Nonlinear dynamics (http://monet.physik.unibas.ch/~elmer/pendulum/ndlDyn.htm)
- Bifurcations and Two Dimensional Flows (http://www.egwald.ca/nonlineardynamics/bifurcations.php) by Elmer G. Wiens
- Introduction to Bifurcation theory (http://prola.aps.org/abstract/RMP/v63/i4/p991_1) by John David Crawford
Saddle-node bifurcation

In the mathematical area of bifurcation theory a saddle-node bifurcation, tangential bifurcation or fold bifurcation is a local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other. The term 'saddle-node bifurcation' is most often used in reference to continuous dynamical systems. In discrete dynamical systems, the same bifurcation is often instead called a fold bifurcation. Another name is blue skies bifurcation in reference to the sudden creation of two fixed points.

If the phase space is one-dimensional, one of the equilibrium points is unstable (the saddle), while the other is stable (the node).

The normal form of a saddle-node bifurcation is:

\[
\frac{dx}{dt} = r + x^2
\]

Here \( x \) is the state variable and \( r \) is the bifurcation parameter.

- If \( r < 0 \) there are two equilibrium points, a stable equilibrium point at \(-\sqrt{-r}\) and an unstable one at \(+\sqrt{-r}\).
- At \( r = 0 \) (the bifurcation point) there is exactly one equilibrium point. At this point the fixed point is no longer hyperbolic. In this case the fixed point is called a saddle-node fixed point.
- If \( r > 0 \) there are no equilibrium points.

A saddle-node bifurcation occurs in the consumer equation (see transcritical bifurcation) if the consumption term is changed from \( px \) to \( p \), that is the consumption rate is constant and not in proportion to resource \( x \).

Saddle-node bifurcations may be associated with hysteresis loops and catastrophes.

Example

An example of a saddle-node bifurcation in two-dimensions occurs in the two-dimensional dynamical system:

\[
\begin{align*}
\frac{dx}{dt} &= \alpha - x^2 \\
\frac{dy}{dt} &= -y.
\end{align*}
\]

As can be seen by the animation obtained by plotting phase portraits by varying the parameter \( \alpha \).

- When \( \alpha \) is negative, there are no equilibrium points.
- When \( \alpha = 0 \), there is a saddle-node point.
- When \( \alpha \) is positive, there are two equilibrium points: that is, one saddle point and one node (either an attractor or a repellor).
transcritical bifurcation

In bifurcation theory, a field within mathematics, a transcritical bifurcation is a particular kind of local bifurcation, meaning that it is characterized by an equilibrium having an eigenvalue whose real part passes through zero.

A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied.\(^1\) In other words, both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide. So the unstable fixed point becomes stable and vice versa.

The normal form of a transcritical bifurcation is

\[
\frac{dx}{dt} = rx - x^2.
\]

This equation is similar to logistic equation but in this case we allow \( r \) and \( x \) to be positive or negative (while in the logistic equation \( x \) and \( r \) must be non-negative). The two fixed points are at \( x = 0 \) and \( x = r \). When the parameter \( r \) is negative, the fixed point at \( x = 0 \) is stable and the fixed point \( x = r \) is unstable. But for \( r > 0 \), the point at \( x = 0 \) is unstable and the point at \( x = r \) is stable. So the bifurcation occurs at \( r = 0 \).

A typical example (in real life) could be the consumer-producer problem where the consumption is proportional to the (quantity of) resource.

For example:

\[
\frac{dx}{dt} = rx(1 - x) - px,
\]

where

- \( rx(1 - x) \) is the logistic equation of resource growth; and
- \( px \) is the consumption, proportional to the resource \( x \).

References

Pitchfork bifurcation

In bifurcation theory, a field within mathematics, a pitchfork bifurcation is a particular type of local bifurcation. Pitchfork bifurcations, like Hopf bifurcations have two types - supercritical or subcritical.

In continuous dynamical systems described by ODEs—i.e. flows—pitchfork bifurcations occur generically in systems with symmetry.

Supercritical case

The normal form of the supercritical pitchfork bifurcation is

$$\frac{dx}{dt} = r x - x^3.$$  

For negative values of $r$, there is one stable equilibrium at $x = 0$. For $r > 0$ there is an unstable equilibrium at $x = 0$, and two stable equilibria at $x = \pm \sqrt{r}$.

Subcritical case

The normal form for the subcritical case is

$$\frac{dx}{dt} = r x + x^3.$$  

In this case, for $r < 0$ the equilibrium at $x = 0$ is stable, and there are two unstable equilibria at $x = \pm \sqrt{-r}$. For $r > 0$ the equilibrium at $x = 0$ is unstable.

Formal definition

An ODE

$$\dot{x} = f(x, r)$$  

described by a one parameter function $f(x, r)$ with $r \in \mathbb{R}$ satisfying:

$$-f(x, r) = f(-x, r) \quad (f \text{ is an odd function}),$$  

$$\frac{\partial f}{\partial x}(0, r_o) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0, r_o) = 0, \quad \frac{\partial^3 f}{\partial x^3}(0, r_o) \neq 0,$$  

$$\frac{\partial f}{\partial r}(0, r_o) = 0, \quad \frac{\partial^2 f}{\partial r \partial x}(0, r_o) \neq 0.$$
has a **pitchfork bifurcation** at \((x, \tau) = (0, \tau_0)\). The form of the pitchfork is given by the sign of the third derivative:

\[
\frac{\partial^3 f}{\partial x^3}(0, \tau_0) \begin{cases} < 0, & \text{supercritical} \\ > 0, & \text{subcritical} \end{cases}
\]

**References**


**Period-doubling bifurcation**

In mathematics, a **period doubling bifurcation** in a discrete dynamical system is a bifurcation in which the system switches to a new behavior with twice the period of the original system. That is, there exists two points such that applying the dynamics to each of the points yields the other point. Period doubling bifurcations can also occur in continuous dynamical systems, namely when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one.

**Examples**

- Logistic map
- Logistic map for a modified Phillips curve

Consider the following logistical map for a modified Phillips curve:

\[
\begin{align*}
\pi_t &= f(u_t) + a \pi_t^e \\
\pi_{t+1} &= \pi_t^e + c(\pi_t - \pi_t^e) \\
f(u) &= \beta_1 + \beta_2 e^{-u}
\end{align*}
\]

where \(b > 0, 0 \leq c \leq 1, \frac{df}{du} < 0\)

where \(\pi\) is the actual inflation, \(\pi^e\) is the expected inflation, \(u\) is the level of unemployment, and \(m - \pi\) is the money supply growth rate. Keeping \(\beta_1 = -2.5, \beta_2 = 20, c = 0.75\) and varying \(b\), the system undergoes period doubling bifurcations, and after a point becomes chaotic, as illustrated in the bifurcation diagram on the right.

- Complex quadratic map
Period-halving bifurcation

A period halving bifurcation in a dynamical system is a bifurcation in which the system switches to a new behavior with half the period of the original system. A series of period-halving bifurcations leads the system from chaos to order.

References


External links

- The Flip (Period Doubling) Bifurcation[^1] in Discrete Time, Dynamic Processes

References

[^1]: http://www egwald com/nonlineardynamics/onedimensionaldynamics_1 php#flipbifurcationconditions
Hopf bifurcation

In the mathematical theory of bifurcations, a Hopf or Poincaré–Andronov–Hopf bifurcation, named after Henri Poincaré, Eberhard Hopf, and Aleksandr Andronov, is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small-amplitude limit cycle branching from the fixed point.

For a more general survey on Hopf bifurcation and dynamical systems in general, see.\cite{1}\cite{2}\cite{3}\cite{4}\cite{5}

Overview

Supercritical / subcritical Hopf bifurcations

The limit cycle is orbitally stable if a specific quantity called the first Lyapunov coefficient is negative, and the bifurcation is supercritical. Otherwise it is unstable and the bifurcation is subcritical.

The normal form of a Hopf bifurcation is:
\[
\frac{dz}{dt} = z((\lambda + i) + b|z|^2),
\]
where \(z, b\) are both complex and \(\lambda\) is a parameter. Write
\[
b = \alpha + i\beta.
\]
The number \(\alpha\) is called the first Lyapunov coefficient.

- If \(\alpha\) is negative then there is a stable limit cycle for \(\lambda > 0\):
  \[
z(t) = re^{i\omega t}
  \]
  where
  \[
r = \sqrt{-\lambda/\alpha} \text{ and } \omega = 1 + \beta r^2.
  \]
The bifurcation is then called supercritical.

- If \(\alpha\) is positive then there is an unstable limit cycle for \(\lambda < 0\). The bifurcation is called subcritical.

Remarks

The "smallest chemical reaction with Hopf bifurcation" was found in 1995 in Berlin, Germany.\cite{6} The same biochemical system has been used in order to investigate how the existence of a Hopf bifurcation influences our ability to reverse-engineer dynamical systems.\cite{7}

Under some general hypothesis, in the neighborhood of a Hopf bifurcation, a stable steady point of the system gives birth to a small stable limit cycle. Remark that looking for Hopf bifurcation is not equivalent to looking for stable limit cycles. First, some Hopf bifurcations (e.g. subcritical ones) do not imply the existence of stable limit cycles; second, there may exist limit cycles not related to Hopf bifurcations.
Example

Hopf bifurcations occur in the Hodgkin–Huxley model for nerve membrane (no citation), the Selkov model\(^{[8]}\) of glycolysis, the Belousov–Zhabotinsky reaction, the Lorenz attractor and in the following simpler chemical system called the Brusselator as the parameter \(B\) changes:

\[
\begin{align*}
\frac{dX}{dt} &= A + X^2Y - (B + 1)X \\
\frac{dY}{dt} &= BX - X^2Y.
\end{align*}
\]

The Selkov model is

\[
\frac{dx}{dt} = -x + ay + x^2y,
\]
\[
\frac{dy}{dt} = b - ay - x^2y.
\]

The phase portrait illustrating the Hopf bifurcation in the Selkov model is shown on the right. See Strogatz, Steven H. (1994). "Nonlinear Dynamics and Chaos",\(^{[1]}\) page 205 for detailed derivation.

Definition of a Hopf bifurcation

The appearance or the disappearance of a periodic orbit through a local change in the stability properties of a steady point is known as the Hopf bifurcation. The following theorem works with steady points with one pair of conjugate nonzero purely imaginary eigenvalues. It tells the conditions under which this bifurcation phenomenon occurs.

**Theorem** (see section 11.2 of\(^{[3]}\)). Let be the Jacobian of a continuous parametric dynamical system evaluated at a steady point \(Z_0\) of it. Suppose that all eigenvalues of have negative real parts except one conjugate nonzero purely imaginary pair \(\pm i\beta\). A Hopf bifurcation arises when these two eigenvalues cross the imaginary axis because of a variation of the system parameters.

Routh–Hurwitz criterion

Routh–Hurwitz criterion (section I.13 of\(^{[5]}\)) gives necessary conditions so that a Hopf bifurcation occurs. Let us see how one can use concretely this idea.\(^{[9]}\)

Sturm series

Let \(p_0, p_1, \ldots, p_k\) be Sturm series associated to a characteristic polynomial \(P\). They can be written in the form:

\[
p_i(\mu) = c_{i,0}\mu^{k-i} + c_{i,1}\mu^{k-i-2} + c_{i,2}\mu^{k-i-4} + \ldots
\]

The coefficients \(c_{i,j}\) for \(i\) in \(\{1, \ldots, k\}\) correspond to what is called Hurwitz determinants.\(^{[9]}\) Their definition is related to the associated Hurwitz matrix.
Propositions

**Proposition 1.** If all the Hurwitz determinants $c_i$ are positive, apart perhaps then the associated Jacobian has no pure imaginary eigenvalues.

**Proposition 2.** If all Hurwitz determinants $c_i$ for all $i$ in $\{0, \ldots, k - 2\}$ are positive, $c_{k-1,0} = 0$ and $c_{k-2,1} < 0$ then all the eigenvalues of the associated Jacobian have negative real parts except a purely imaginary conjugate pair.

The conditions that we are looking for so that a Hopf bifurcation occurs (see theorem above) for a parametric continuous dynamical system are given by this last proposition.

Example

Let us consider the classical Van der Pol oscillator written with ordinary differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= \mu(1 - y^2)x - y, \\
\frac{dy}{dt} &= x.
\end{align*}
\]

The Jacobian matrix associated to this system follows:

\[
J = \begin{pmatrix}
-\mu(-1 + y^2) & -2\mu y x - 1 \\
1 & 0
\end{pmatrix}.
\]

The characteristic polynomial (in $\lambda$) of the linearization at $(0,0)$ is equal to:

\[P(\lambda) = \lambda^2 - \mu\lambda + 1.\]

The coefficients are: $a_0 = 1, a_1 = -\mu, a_2 = 1$.

The associated Sturm series is:

\[
p_0(\mu) = a_0 \lambda^2 - a_2 \\
p_1(\lambda) = a_1 \lambda
\]

The Sturm polynomials can be written as (here $i = 0, 1$):

\[p_i(\mu) = c_{i,0}\mu^{k-i} + c_{i,1}\mu^{k-i-1} + c_{i,2}\mu^{k-i-2} + \ldots\]

The above proposition 2 tells that one must have:

\[c_{0,0} = 1 > 0, c_{1,0} = -\mu = 0, c_{0,1} = -1 < 0.\]

Because $1 > 0$ and $-1 < 0$ are obvious, one can conclude that a Hopf bifurcation may occur for Van der Pol oscillator if $\mu = 0$.

References


External links

- Reaction-diffusion systems
- The Hopf Bifurcation (http://www.egwald.com/nonlineardynamics/bifurcations.php#hopfbifurcation)
- Andronov–Hopf bifurcation page (http://www.scholarpedia.org/article/Andronov-Hopf_bifurcation) at Scholarpedia
Global

Homoclinic bifurcation

In mathematics, a **homoclinic bifurcation** is a global bifurcation which often occurs when a periodic orbit collides with a saddle point.

The image below shows a phase portrait before, at, and after a homoclinic bifurcation in 2D. The periodic orbit grows until it collides with the saddle point. At the bifurcation point the period of the periodic orbit has grown to infinity and it has become a homoclinic orbit. After the bifurcation there is no longer a periodic orbit.

A homoclinic bifurcation occurs when a periodic orbit collides with a saddle point. **Left panel:** For small parameter values, there is a saddle point at the origin and a limit cycle in the first quadrant. **Middle panel:** As the bifurcation parameter increases, the limit cycle grows until it exactly intersects the saddle point, yielding an orbit of infinite duration. **Right panel:** When the bifurcation parameter increases further, the limit cycle disappears completely.

Homoclinic bifurcations can occur supercritically or subcritically. The variant above is the "small" or "type I" homoclinic bifurcation. In 2D there is also the "big" or "type II" homoclinic bifurcation in which the homoclinic orbit "traps" the other ends of the unstable and stable manifolds of the saddle. In three or more dimensions, higher codimension bifurcations can occur, producing complicated, possibly chaotic dynamics.
Heteroclinic bifurcation

In mathematics, particularly dynamical systems, a **heteroclinic bifurcation** is a global bifurcation involving a heteroclinic cycle. Heteroclinic bifurcations come in two types, resonance bifurcations, and transverse bifurcations. Both types of bifurcation will result in the change of stability of the heteroclinic cycle. At a resonance bifurcation, the stability of the cycle changes when an algebraic condition on the eigenvalues of the equilibria in the cycle is satisfied. This is usually accompanied by the birth or death of a periodic orbit. A transverse bifurcation of a heteroclinic cycle is caused when the real part of a transverse eigenvalue of one of the equilibria in the cycle passes through zero. This will also cause a change in stability of the heteroclinic cycle.

**References**


Infinite-period bifurcation

In mathematics, an **infinite-period bifurcation** is a global bifurcation that can occur when two fixed points emerge on a limit cycle. As the limit of a parameter approaches a certain critical value, the speed of the oscillation slows down and the period approaches infinity. The infinite-period bifurcation occurs at this critical value. Beyond the critical value, the two fixed points emerge continuously from each other on the limit cycle to disrupt the oscillation and form two saddle points.

**References**

Blue sky catastrophe

The **blue sky catastrophe** is a type of bifurcation of a periodic orbit. In other words, it describes a sort of behaviour stable solutions of a set of differential equations can undergo as the equations are gradually changed. This type of bifurcation is characterised by both the period and length of the orbit approaching infinity as the control parameter approaches a finite bifurcation value, but with the orbit still remaining within a bounded part of the phase space, and without loss of stability before the bifurcation point. In other words, the orbit *vanishes into the blue sky*.

The bifurcation has found application in, amongst other places, slow-fast models of computational neuroscience. The possibility of the phenomenon was raised by David Ruelle and Floris Takens in 1971, and explored by R.L. Devaney and others in the following decade. More compelling analysis was not performed until the 1990s.

This bifurcation has also been found in the context of fluid dynamics, namely in double-diffusive convection of a small Prandtl number fluid. Double diffusive convection occurs when convection of the fluid is driven by both thermal and concentration gradients, and the temperature and concentration diffusivities take different values. The bifurcation is found in an orbit that is born in a global saddle-loop bifurcation, becomes chaotic in a period doubling cascade, and disappears in the blue sky catastrophe.

**References**

- Blue Sky Catastrophe[^1] article in Scholarpedia
- Andrey Shilnikov[^2] - studies the blue sky catastrophe and other topics in dynamical neuroscience.

**References**

[^1]: http://www.scholarpedia.org/article/Blue-Sky_Catastrophe
[^2]: http://www2.gsu.edu/~matals/
[^3]: http://prola.aps.org/abstract/PRL/v92/i23/e234501
Article Sources and Contributors


Image Sources, Licenses and Contributors


